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COMMENT

Scaling relation for the critical exponents of the backbone of percolation clusters

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Abstract. We suggest that β_B, the critical exponent of the backbone of percolation clusters, is given by $\beta_B = \frac{1}{2}(\gamma + 5\beta) - 1$, where γ and β are susceptibility and order parameter exponents of percolation respectively. The proposed relation agrees with the ε expansion of β_B given by Harris and Lubensky and provides accurate estimates of β_B at all $d > 1$. This relation may suggest that the backbone exponents *are not* independent of other percolation exponents. We also suggest a more general relation for the order parameter exponent of m-connected clusters.

Although it was hoped in the pioneering work of Broadbent and Hammersley (1957) that percolation theory would find applications in the problems of fluid flow in porous media, this idea was not developed to a large extent for many years. Only recently has some progress been made and concepts of percolation theory have been utilised to describe flow of fluids in porous media. In particular, percolation theory has been employed to explain mixing of two fluids in an unsaturated porous medium (Sahimi 1984a, Sahimi *et al* 1983), to describe how several immiscible fluid phases distribute themselves in a porous medium (Heiba *et al* 1982, 1984a, b) and how two immiscible phases displace one another (Chandler *et al* 1982, Wilkinson and Willemsen 1983). One of the main problems in such modelling is that only the multiply connected part of the infinite cluster supports the transport process; the rest of the cluster is the dead-end or dangling part. The biconnected part of the cluster is called the backbone (Kirkpatrick 1978). A new exponent β_B (sometimes called β') was introduced by Kirkpatrick (1978) to describe the fraction $B(p)$ of total sites (bonds) that belong to the backbone:

$$B(p) \sim (p - p_c)^{\beta_B}. \tag{1}$$

Here p is the fraction of active sites (bonds) and p_c is the percolation threshold. In analogy with the critical exponents that describe the statistics of percolation clusters, one can also define the susceptibility exponent γ_B, correlation length exponent ν_B and so on.

One of the main questions has been whether the critical exponents of the backbone are totally independent exponents, or are related to the traditional percolation exponents. Shlifer *et al* (1979) showed that the correlation length exponent is the same

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for both the percolation clusters and the backbone, i.e. $\nu = \nu_B$. The backbone of the largest percolation cluster at p_c is recognised to be a fractal object with a fractal dimensionality d_B which is given by (Kirkpatrick 1978)

$$d_B = d - \beta_B/\nu, \tag{2}$$

where d is the dimensionality of the system. We compile in table 1 the available numerical estimates for the backbone exponents. The results of Li and Strieder (1982a, b) were obtained by the simple Monte Carlo method, while those of Kirkpatrick (1978) were obtained by Monte Carlo simulations and finite-size scaling arguments. Puech and Rammal (1983) made direct measurements of d_B, while Shlifer *et al* (1979) employed a position-space renormalisation group method. Hong and Stanley (1983) developed low-density series expansions to estimate β_B and d_B. Finally, Herrmann *et al* (1984) made a direct measurement of d_B by the new method of 'burning' introduced by Stauffer (1984). These are the only data we are aware of.

Table 1. Comparison of predicted values of backbone exponent β_B and its fractal dimensionality d_B with the available data.

d	β	ν	β_B (predicted)	β_B (data)	d_B (predicted)	d_B (data)
				0.43 ± 0.02[b]		1.63 ± 0.01[d]
2	$\frac{5}{36}$	$\frac{4}{3}$	$39/72 \simeq 0.542$[a]	0.38 ± 0.02[c]	$153/96 \simeq 1.594$[a]	1.68 ± 0.02[f]
				0.50 ± 0.02[d]		1.60 ± 0.05[g]
				0.50–0.60[e]		1.80 ± 0.04[h]
3	0.42	0.88	0.960	0.9 ± 0.1[e]	1.910	1.77 ± 0.07[g]
						1.83 ± 0.08[h]
4	0.62	0.66	1.250	1.1 ± 0.2[e]	2.106	$1.89\,{}^{+0.14}_{-0.21}$[h]
5	0.84	0.57	1.685	—	2.044	1.93 ± 0.16[h]
$\geqslant 6$	1	$\frac{1}{2}$	2	2[i]	2	2

[a] This is the predicted value if equation (5) is valid at $d = 2$. [b] Li and Strieder (1982a). [c] Li and Strieder (1982b). [d] Shlifer *et al* (1979). [e] Kirkpatrick (1978). [f] Puech and Rammal (1983). [g] Herrmann *et al* (1984). [h] Hong and Stanley (1983). [i] Larson and Davis (1982).

Harris (1983) and Harris and Lubensky (1983) (hereafter referred to as HL) have recently developed a field-theoretic approach to the backbone problem and derived an ε expansion ($\varepsilon = 6 - d$) for the backbone exponents. In particular, they have shown that

$$\beta_B = 2\beta + \nu\psi^{(2)}, \tag{3}$$

where β is the critical exponent of percolation probability and $\psi^{(2)}$ is a crossover exponent describing the correlation function of the backbone. According to these authors $\psi^{(2)}$ is an independent exponent and thus the critical exponents of the backbone *cannot* be described in terms of the exponents of the infinite cluster. They also obtained to order ε^2, $\psi^{(2)} = 2\varepsilon^2/49$, which then yields

$$\beta_B = 2 - 2\varepsilon/7 + 65\varepsilon^2/6174 + \ldots. \tag{4}$$

This result means that $d_B = 2 + \varepsilon/21 + \ldots$, i.e. d_B has a non-monotonic dependence on d.

In this paper we present a simple relation which relates β_B to β and ν. The proposed relation agrees with the ε expansion of β_B, equation (4). This agreement suggests strongly that the proposed equation may be exact, although we do not have any rigorous proof. Therefore, if our scaling relation is exact, *all* backbone critical exponents can be described in terms of those of the infinite cluster and thus $\psi^{(2)}$ is *not* an independent exponent. The proposed scaling relation is given by

$$\beta_B = \tfrac{1}{2}(\nu d + 3\beta) - 1. \tag{5}$$

If we use (Amit 1976, Priest and Lubensky 1976) $\beta = 1 - \varepsilon/7 - 61\varepsilon^2/12348$ and $\nu = \frac{1}{2} + 5\varepsilon/84 + 589\varepsilon^2/37044$ in equation (5), the result agrees with equation (4). This calls for the calculation of higher-order terms of the ε expansion of $\psi^{(2)}$ to assess further the validity of equation (5). The plausible guess (Kirkpatrick 1978) $\beta_B = 2\beta$ breaks down at ε^2 order. Equation (5) can also be written as $\beta_B = \frac{1}{2}(\gamma + 5\beta) - 1$, so that it will also be valid above six dimensions, the upper critical dimensionality of percolation.

In table 1 we list the predictions of equation (5) for β_B; we also calculate the fractal dimensionality d_B of the backbone, the results of which are also listed in table 1. Equation (5) correctly predicts that d_B is a non-monotonic function of d. Equation (5) is not valid at $d = 1$ where one expects to have $\beta_B = 0$. HL pointed out that their field-theoretic derivation of (3) breaks down at a dimensionality d_l which they estimated to be $d_l \simeq 3$. In a previous paper (Sahimi 1984b) we expressed the opinion that d_l is the dimensionality at which $d_f = 2$, where $d_f = d - \beta/\nu$ is the fractal dimensionality of the largest percolation cluster at p_c; this then means that $d_l \simeq 2.2$. From equations (3) and (5) we obtain $\nu\psi^{(2)} = \Delta/2 - 1$, where $\Delta = \beta + \gamma$ is the 'gap' exponent. Since $\nu\psi^{(2)} > 0$, we must have $\Delta > 2$, i.e. $d > 1.65$. But we suggest that equation (5) may be valid for $d_f \geqslant 2$, i.e. for $\Delta/\nu \geqslant 2$. If we assume that (5) is also valid at $d = 2$, we obtain $\beta_B(d = 2) = 39/72 \simeq 0.541$. This is consistent with the result of Kirkpatrick (1978), $0.5 < \beta_B(d = 2) < 0.6$, and not too much larger than the estimate of Shlifer *et al* (1979), $\beta_B(d = 2) = 0.50 \pm 0.02$. The predicted value of $\beta_B(d = 2)$ also implies that $d_B(d = 2) = 153/72 \simeq 1.594$, in excellent agreement with the estimate of Herrmann *et al* (1984), $d_B = 1.60 \pm 0.05$. It is also not too different from other estimates of d_B (see table 1). The work of HL also indicates that the backbone exponents obey the usual scaling and hyperscaling laws, e.g. $2\beta_B + \gamma_B = \nu d$. Thus we obtain a simple equation for γ_B:

$$\gamma_B = 2 - 3\beta. \tag{6}$$

Equation (5) is certainly much more accurate in estimating d_B than the equation $d_B = \ln(d+1)/\ln 2$, which is the prediction of the Sierpinski gasket model of the backbone (Gefen *et al* 1981). It also provides more accurate estimates of d_B than the present series estimates of Hong and Stanley (1983).

One can more generally study m-connected clusters in which two sites which are widely separated are connected to each other by m independent paths of active sites. One can define, in a similar manner, the order parameter $\beta^{(m)}$ for these clusters so that $\beta^{(1)} = \beta$ and $\beta^{(2)} = \beta_B$. Similarly, the correlation function of the m-connected clusters is described by an exponent $\psi^{(m)}$.

According to the work of HL one has, to order ε^2, $\psi^{(m)} = m(m-1)\varepsilon^2/49$ and one also has $\beta^{(m)} = m\beta + \nu\psi^{(m)}$. We propose that

$$\beta^{(m)} = \tfrac{1}{4}[(m^2 - m)\nu d + (5m - m^2)\beta] - \tfrac{1}{2}(m^2 - m), \tag{7}$$

which reduces to (5) for $m = 2$. This equation too agrees with the ε expansion of $\beta^{(m)}$ given by HL. In a previous paper (Sahimi 1984b) we observed that the percolation equation (5), we obtain $t = (\nu d + 3\beta)/2$. This relation for t is similar in its dependence on ν and β to the Alexander–Orbach conjecture (Alexander and Orbach 1982), $t = [(3d-4)\nu - \beta]/2$. These two relations yield very similar results for $d \geqslant d_l$, although they cannot be equal for general d. Even if our proposed equations (5) and (7) turn out to be only approximate formulae, they are certainly much more accurate than any Flory-like approximations which usually break down in order ε. For example, a Flory approximation for d_f, the fractal dimension of the largest percolation cluster at p_c, yields (Isaacson and Lubensky 1980) $d_f = d/2 + 1$, i.e. $d_f = 4 - \varepsilon/2$, in complete disagreement with the results of Amit (1976) and Priest and Lubensky (1976).

With the aid of equation (5), which can be written as $\beta_B = (\gamma + 5\beta)/2 - 1$, so that it will also be valid above six dimensions, one can obtain accurate estimates for $\bar{d}_s$, the spectral dimension of the backbone. $\bar{d}_s$ is given by $\bar{d}_s = 2(\nu d - \beta_B)/(2\nu + t - \beta_B)$. The results are also listed in table 1. These results show that the spectral dimension of the backbone varies continuously between $d = 2$ and $d = 6$, in contrast with the spectral dimension of the largest percolation cluster at p_c which remains approximately constant.

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